Real Time Linear Scheduling

Fast Approximation Solution to the Assignment Problem

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Algo Trading Motivation

In a response to market event, we need to “fire” batch of market orders via a single pipe

What’s the optimal ordering if we want to maximize cumulative fill rate?

The Linear Scheduling Problem

- We need to execute a set of tasks in a sequential manner
- We want to solve the optimal execution order
- Task value is dependent on the time we execute, e.g. the further we delay the execution time, the less value we will extract from from the execution
- We want to make the ordering decision as fast as possible

Assignment Problem Formalization:

Given $V_i(t)$ - the execution value of i-th task when executed in the t-th slot
Solve

$$\arg\max_{\pi \in S_n} \sum_i V_i(\pi(i))$$

Assignment Problem Solvers

Linear Programming

- Define continuous variables $x_{i,j} \geq 0$
- Maximize $$\max_{x_{i,j}} \sum_{i,j} x_{i,j} V_i(j)$$
- Such that $$\sum_i x_{i,j} = 1 \text{ and } \sum_j x_{i,j} = 1$$
- Since LP solution is obtained on the simplex vertices, solution will be integral.

Khun (1955) presented a combinatorial algorithm named *"The Hungurain Method"*

- Minimize cost of a perfect matching in a complete bipartite graph
- Iteratively augment matching and vertices potential function, until cost gap between potential and matching is closed.

Munkres (1957) proved the complexity is $O(n^4)$

Edmons and Karp later improved the algorithm to $O(n^3)$.

Can We Do It Faster ?

For scheduling problem:

We can expect the value functions V_i to be continuous decreasing functions of time

If each V_i was linear in time, then optimal ordering was according to slopes

Thus, first order approximation: sort according to $\frac{\partial V_i}{\partial t}\left(\frac{n+1}{2}\right)$ as t in {1,..,n}

Let's try to do better using continuous relaxation.

Continuous Relaxation

Instead of solving permutation π define continuous variables $x_i = \pi(i) - \frac{n+1}{2}$

To ease calculation let's define also $G_i(x) = V_i(x + \frac{n+1}{2})$

Since π is a permutation we can impose the following constraints $\sum_i x_i = 0$ and $\sum_i x_i^2 = \frac{n^3}{12}$

when maximizing $\sum_i G_i(x_i)$

Define the lagrangian $\mathcal{L}(x_i, \lambda, \gamma) = \sum_i G_i(x_i) - \lambda \sum_i x_i - \gamma \sum_i x_i^2$

Solving for zero partial derivative we get $G_i'(x_i) = \lambda + 2\gamma x_i$

and playing a bit with the constraints we get also

$\lambda = \frac{1}{n}\sum_i G_i'(x_i) = E(\{G_i'(x_i)\})$ $\gamma^2 = \frac{3}{n^3}\sum_i (G_i'(x_i) - \lambda)^2 = \frac{3}{n^2}\,Var(\{G_i'(x_i)\})$

Proposed Scheme

Initial Point

Define $\{x_i\}$ according to the rank of the value functions derivative

$$x_i = \mathrm{rank}_i\{G_i'(0)\} - \frac{n+1}{2}$$

Ping Pong Solver

Given $\{x_i\}$ define

$$\lambda = E(\{G_i'(x_i)\}) \quad \gamma^2 = \frac{3}{n^2} Var(\{G_i'(x_i)\})$$

Given $\{\lambda, \gamma\}$ do a line search per x_i to solve

$$G_i'(x_i) = \lambda + 2\gamma x_i$$

Repeat until fixed point is obtained

Transform Result

Rank to obtain the scheduling order

$$\pi(i) = \mathrm{rank}_i(\{x_i\})$$

Notes

There might be multiple solutions for $G_i'(x_i) = \lambda + 2\gamma x_i$. Since we are looking for a maximum we can add the condition $G_i''(x_i) < 2\gamma$ to pick one.

We can approximate the value functions by low order polynomials (e.g. using taylor expansion) and use a closed formula solution for $G_i'(x_i) = \lambda + 2\gamma x_i$, saving the line search phase.

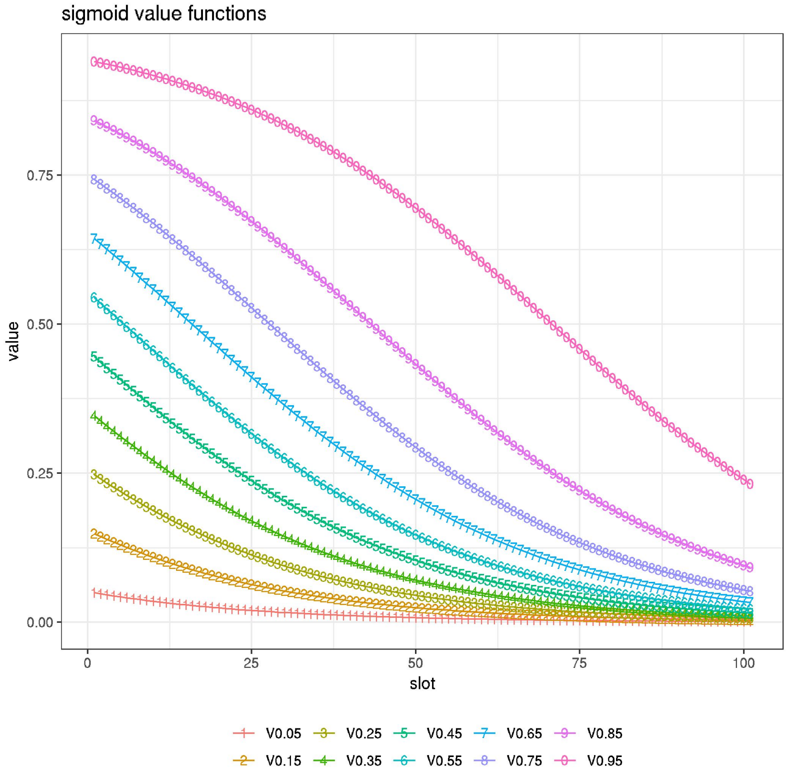
We can impose additional constraints on $\{x_i\}$ making them closer to permutation, for example

$$\sum_i x_i^3 = 0 \quad \text{and} \quad \sum_i x_i^4 = \frac{n^5}{80}$$

Example - Sigmoid Value Functions

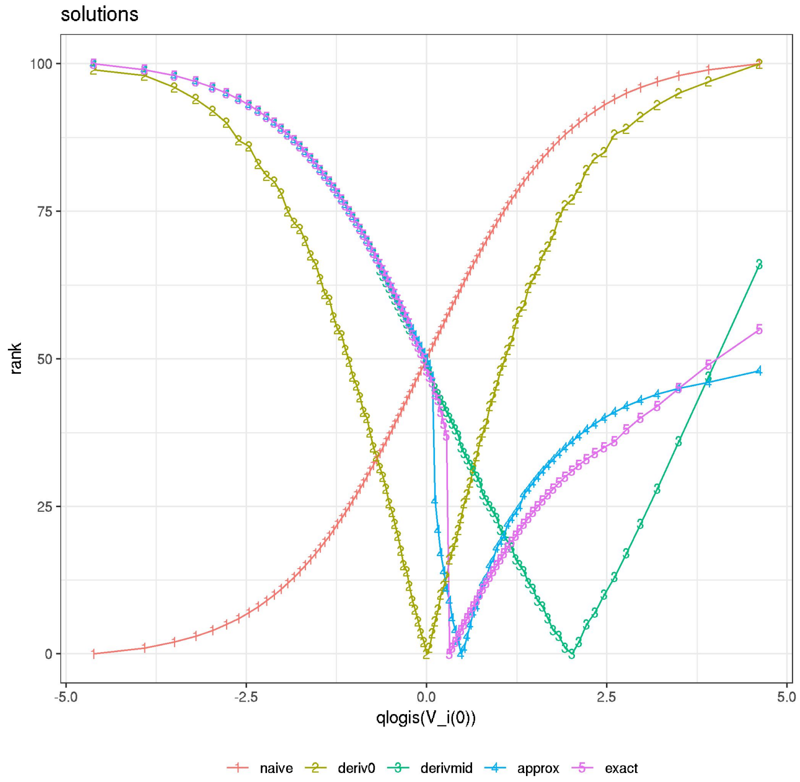
$$V_i(t) = \sigma(a_i - b\,t) = \sigma\left(\sigma^{-1}\left(\frac{i}{n+1}\right) - \frac{4(t-1)}{n}\right)$$

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



Example - Sigmoid Value Functions

```
Scores
    naive    deriv0  derivmid    approx     exact
0.1359592 0.2316456 0.2958575 0.3059043 0.3071864
```



Other Applications

- Real-time Advertising Bidding
- Network Routing Decisions
- Speed Dating ?